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ON THE LIMIT CASE OF ENDOCHRONIC THEORY

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Abstract—An endochronic model is described and shown to lead to elastic as well as plastic behavior in the limit case of $k = 1$, in contrast to Fazio's claim that the limit case cannot lead to plasticity. The slope of the stress–strain curve is represented by a function $F(X)$ whose functional form is numerically investigated. The study includes straining, unstraining and cyclic straining. The equations of the general form of this model have also been derived.

1. INTRODUCTION

A modified endochronic theory was presented by Valanis (1980). The modification was based on an intrinsic time defined by a plastic-strain-like tensor, which in the one-dimensional case is given by

$$dQ = d\varepsilon - k\frac{d\sigma}{E} \tag{1}$$

where σ and ε are stress and strain, respectively; Q is the plastic-strain-like quantity; E is Young's modulus; and k is a constant parameter having a value between zero and one. In the case of $k = 0$, the 1980 theory reduces to the earlier theory of Valanis (1971). In the case of $k = 1$, Valanis (1980) shows that the 1980 theory led to the classical theory of plasticity with yield surface defined and subjected to a combined isotropic–kinematic strain-hardening rule. The normality condition of the plastic strain rate with respect to the yield surface was also derived from this theory. The plastic-strain-like increment dQ reduces to the plastic strain increment $d\varepsilon^p$ when $k = 1$, and is given by

$$d\varepsilon^p = d\varepsilon - \frac{d\sigma}{E}. \tag{2}$$

In the treatment of the case of $k = 1$, Valanis (1980) cautioned that care must be exercised in the derivation, otherwise erroneous results might be obtained.

In a recent paper, Fazio (1989), in a one-dimensional study of the limit case of the endochronic theory, claimed that, in the case of $k = 1$, the endochronic theory could only lead to elastic behavior and that plastic behavior could not develop according to the 1980 theory. The purpose of this paper is to show that Fazio's claim is not valid. Furthermore, additional details for the case of $k = 1$ are presented. The one-dimensional case is first discussed, and a discussion of the three-dimensional case follows. In this paper, explicit equations using r number of internal variables have been derived.

2. ONE-DIMENSIONAL ENDOCHRONIC CONSTITUTIVE EQUATION

Following the 1980 theory, the constitutive equation in the Gibbs formulation (see Wu and Aboutorabi, 1988) is given by

$$d\varepsilon = \frac{d\sigma}{E} + \sum_{i=1}^{r} C_i\, dq^i \tag{3}$$

where q^i are r number of internal variables, and C_i are material constants (see the Appendix for additional discussion concerning the Helmholtz and Gibbs formulation). The internal variables q^i may be interpreted as the descriptors for the internal structure of a material during plastic deformation, and these variables evolve with respect to an intrinsic time z, which is used to register the deformation history during deformation. The rate of change of the internal variables is determined by the current state of stress and internal variables. In this investigation, a linear evolution equation for the internal variables q^i is assumed such that for each internal variable q^i,

$$\frac{dq^i}{dz} = \frac{1}{N_i}(C_i\sigma - F_i q^i) \quad (i \text{ not summed}), \tag{4}$$

where N_i and F_i are material constants.

In order to account for strain-hardening, the intrinsic time z is scaled by

$$\frac{d\zeta}{dz} = f(z) \tag{5}$$

where the function f represents isotropic hardening. The increment of intrinsic time $d\zeta$ is defined in terms of the plastic-strain-like quantity, so that

$$d\zeta = |dQ| \tag{6}$$

or

$$d\zeta = \pm\left[d\varepsilon - k\frac{d\sigma}{E}\right] \tag{7}$$

where $d\zeta > 0$.

By combining (4) and (5) it is obtained that

$$dq^i = \frac{d\zeta}{fN_i}(C_i\sigma - F_i q^i) = \frac{X_i}{C_i}\, d\zeta \quad (i \text{ not summed}), \tag{8}$$

where

$$X_i = \frac{C_i}{fN_i}(C_i\sigma - F_i q^i) \quad (i \text{ not summed}). \tag{9}$$

Therefore, by the substitution of (7) and (8), eqn (3) becomes

$$d\varepsilon = \frac{d\sigma}{E} + \sum_{i=1}^{r} X_i\, d\zeta = \frac{d\sigma}{E} \pm X\left[d\varepsilon - k\frac{d\sigma}{E}\right] \tag{10}$$

where

$$X = \sum_{i=1}^{r} X_i. \tag{11}$$

Equation (10) is further written as

$$\frac{d\sigma}{E}[1 \pm kX] = d\varepsilon[1 \pm X]. \tag{12}$$

This equation provides the basis for the discussion to follow.

Fazio (1989), in his eqn (24) (see the Appendix), rewrote the above equation as

$$\frac{d\sigma}{d\varepsilon} = E\left[\frac{1 \pm X}{1 \pm kX}\right] \tag{13}$$

with no mention of the condition

$$1 \pm kX \neq 0. \tag{14}$$

The condition given by (14) should be satisfied in order for (13) to be valid. Fazio went on to say that when $k = 1$, (13) reduces to

$$\frac{d\sigma}{d\varepsilon} = E \tag{15}$$

and has thus arrived at an erroneous conclusion that, in the case of $k = 1$, the endochronic theory could only lead to elastic behavior and that plastic behavior could not develop. A careful observation would lead to the result $1 \pm X = 0$, when $k = 1$. Therefore Fazio, in fact, dealt with the case of 0/0.

The fact that it is possible to deduce the elastic as well as the plastic behavior from the 1980 theory will now be shown. Equation (12) may be rewritten as (13) in the case of $k \neq 1$, because in this case (14) is satisfied. But for the case of $k = 1$, (12) should be rewritten as

$$\left(d\varepsilon - \frac{d\sigma}{E}\right)(1 \pm X) = 0. \tag{16}$$

Thus,

$$d\varepsilon - \frac{d\sigma}{E} = 0 \quad \text{or} \quad 1 \pm X = 0. \tag{17}$$

When $d\varepsilon - d\sigma/E = 0$, the behavior is elastic and generally $X \neq \pm 1$. Hence, the slope of the stress–strain curve is E. But, when $1 \pm X = 0$, i.e. $X = \pm 1$, generally $d\varepsilon - d\sigma/E \neq 0$, and the slope is not E. In fact, the condition $X = \pm 1$ with the help of (9) and (11) leads to

$$X = \sum_{i=1}^{r} \frac{C_i^2}{N_i}\frac{\sigma}{f} - \frac{1}{f}\sum_{i=1}^{r} \frac{C_i F_i q^i}{N_i} = \pm 1, \tag{18}$$

which can then be rewritten as

$$\sigma = \pm\sigma_y f + \sigma_y \sum_{i=1}^{r} \frac{C_i F_i q^i}{N_i} \tag{19}$$

where

$$\sigma_y = \left(\sum_{i=1}^{r} \frac{C_i^2}{N_i}\right)^{-1}. \tag{20}$$

Note that the first term on the right hand side of (19) represents isotropic hardening and the second term the kinematic hardening.

By writing (13) as

$$\frac{d\sigma}{d\varepsilon} = EF(X) \tag{21}$$

where

$$F(X) = \frac{1 \pm X}{1 \pm kX}, \tag{22}$$

the variation of the value of function $F(X)$ may now be investigated. Note that the form of the isotropic hardening function $f(z)$ affects the value of X. The following expression used by Wu and Yip (1981) has been used in the present computation:

$$f(z) = A-(A-1)\exp(-\beta z). \tag{23}$$

A hypothetical material with the following material constants: $N = 3.64$ MPa, $C = 0.54$, $F = 140$ MPa, $A = 3.3$, $\beta = 150$ and $E = 28000$ MPa is now considered to demonstrate the applicability of the present model (with only one internal variable for simplicity) and to show the form of the function $F(X)$. This function has been calculated for $k = 0.5$, 0.95, and 1, and plotted versus X in Fig. 1. It is seen from the figure that, for the cases of $k \neq 1$, the value of $F(X)$ changes gradually with X. But, for the case of $k = 1$, $F(X)$ remains one in the elastic range as X increases; as soon as X reaches one, plastic deformation occurs and $F(X)$ decreases as the plastic deformation accumulates with X staying at one.

The step-by-step numerical procedures are: knowing $d\varepsilon$ and the current values of z, σ, and q, eqn (12) may be used to find $d\sigma$; eqns (1), (23) and (6) can then be used to determine dQ and dz, and eqn (8) leads to dq. Thus z, σ, and q can be updated.

In the case of strain-controlled cyclic loading with strain limits of $\pm 1\%$, the cyclic stress-strain curve is shown in Fig. 2 for this hypothetical material. The corresponding $F(X)$ versus X diagram is shown in Fig. 3. For the case of $k = 0.95$, and similarly for $k = 0.5$, the following sequence is followed: ALCDAMFGALCD.... Note that the change between CD and FG is abrupt; while that between AC and AF is gradual. In the case of k = 1, the following sequence is followed: ABCBAEFEABC.... The change from B to C and from E to F is gradual; while from C to B and F to E is abrupt. The corresponding plot for $F(X)$ versus intrinsic time z is shown in Fig. 4. The letters denote corresponding points in these figures.

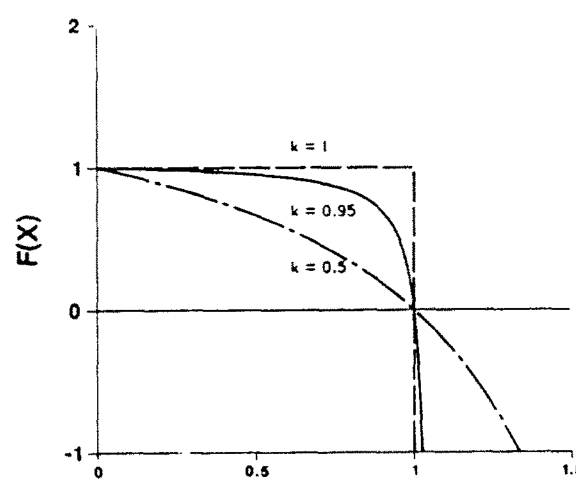

Fig. 1. Function $F(X)$ for various constant k.

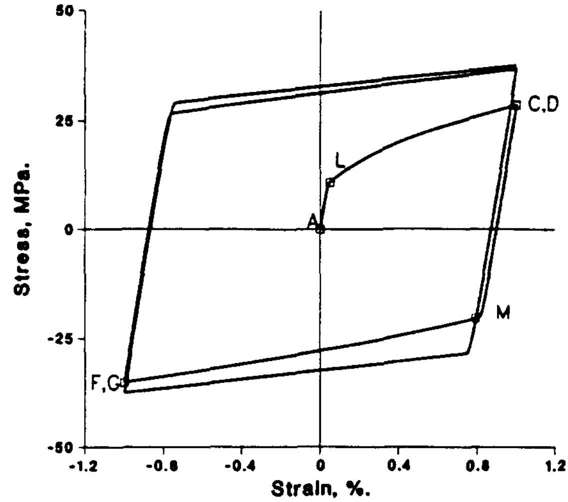

Fig. 2. One-dimensional cyclic stress–strain curve for a hypothetical material, $k = 0.95$.

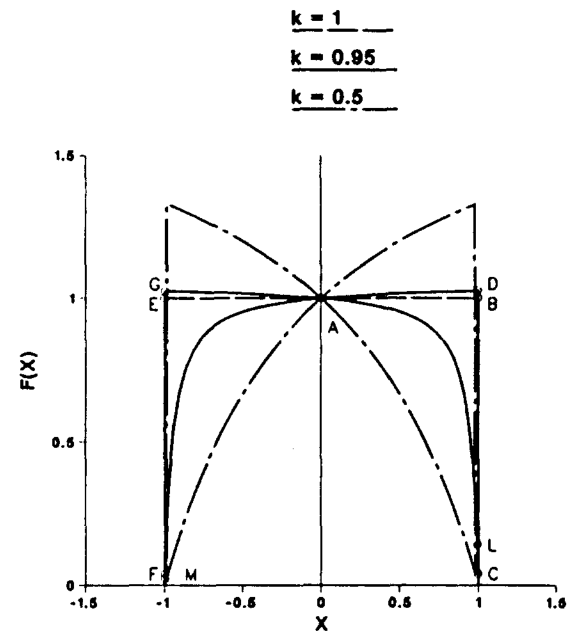

Fig. 3. Function $F(X)$ for cyclic straining.

3. THE THREE-DIMENSIONAL EQUATIONS

In this paper, the material is assumed to be plastically incompressible. For a discussion of material subjected to volumetric plastic deformation, the reader is referred to Wu and Aboutorabi (1988). For plastically incompressible materials, it suffices to consider only the deviatoric response.

In the deviatoric response, the internal variables are $\mathbf{p}^n$, with $n = 1$ to r. The deviatoric strain increment de is expressed in terms of the increments of deviatoric stress ds and internal variables $d\mathbf{p}^n$, i.e.

$$de_{ij} = \frac{ds_{ij}}{2\mu_o} + \sum_{n=1}^{r} C^n \, dp^n_{ij}, \tag{24}$$

where C^n are material constants and μ_o is the shear modulus. The evolution of $\mathbf{p}^n$ is defined

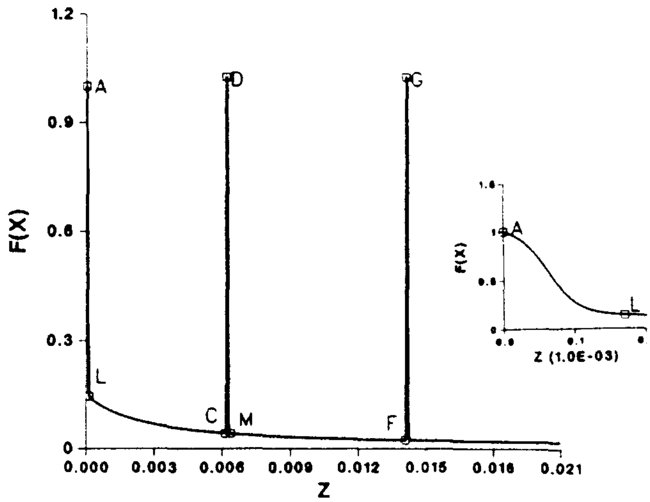

Fig. 4. $F(x)$ versus z for cyclic straining, $k = 0.95$.

with respect to intrinsic time z. By assuming a linear evolution equation, the rate of change of $\mathbf{p}^n$ is given by

$$\frac{\mathrm{d}p_{ij}^n}{\mathrm{d}z} = \left(\frac{C^n}{N^n} s_{ij} - \frac{F^n}{N^n} p_{ij}^n\right) \quad (n \text{ not summed}), \tag{25}$$

where N^n, F^n are material constants. A deviatoric strain-like tensor $\mathbf{Q}$ is now defined by

$$\mathrm{d}Q_{ij} = \mathrm{d}e_{ij} - k\frac{\mathrm{d}s_{ij}}{2\mu_0} \tag{26}$$

where k is again a constant parameter such that $0 \leqslant k \leqslant 1$. Note that when $k = 1$, $\mathbf{Q}$ is equal to the plastic strain $\mathbf{e}^p$. An intrinsic time increment $\mathrm{d}\zeta$ is now defined as

$$\mathrm{d}\zeta^2 = \mathrm{d}Q_{ij}\,\mathrm{d}Q_{ij}, \tag{27}$$

which is related to z through the relation given by (5). It will be shown subsequently that function f represents isotropic hardening. In a strain-controlled test, $\mathrm{d}e_{ij}$ is a known and input quantity.

Rewriting eqn (24) as

$$\mathrm{d}s_{ij} = 2\mu_0\left(\mathrm{d}e_{ij} - \sum_{n=1}^{r} C^n\,\mathrm{d}p_{ij}^n\right), \tag{28}$$

this equation together with eqn (25) may be substituted into eqn (26) to yield

$$\mathrm{d}Q_{ij} = (1-k)\,\mathrm{d}e_{ij} + k\sum_{n=1}^{r} C^n\,\mathrm{d}p_{ij}^n \tag{29}$$

or

$$\mathrm{d}Q_{ij} = \alpha_{ij} + \beta_{ij}\,\mathrm{d}z \tag{30}$$

where

$$\alpha_{ij} = (1-k)\,\mathrm{d}e_{ij} \tag{31}$$

$$\beta_{ij} = k\left(\sum_{n=1}^{r} \frac{C^n C^n}{N^n} s_{ij} - \sum_{n=1}^{r} \frac{C^n F^n}{N^n} p_{ij}^n\right). \tag{32}$$

Finally, by substituting eqns (5) and (30) into (27), the following quadratic equation is obtained:

$$P\,\mathrm{d}z^2 + Q\,\mathrm{d}z + R = 0 \tag{33}$$

where

$$P = \beta_{ij}\beta_{ij} - f^2 \tag{34}$$

$$Q = 2\alpha_{ij}\beta_{ij} \tag{35}$$

$$R = \alpha_{ij}\alpha_{ij}. \tag{36}$$

The functions P, Q and R are defined in terms of the constants of the model and increments of input variables. When $\mathrm{d}z$ is solved in eqn (33), eqn (25) can be used to obtain $\mathrm{d}p_{ij}^n$. By substitution, the deviatoric stress increment $\mathrm{d}s_{ij}$ can then be found from eqn (28).

In the case of $k = 1$, eqns (31), (35) and (36) give $\alpha_{ij} = R = Q = 0$. Thus, eqn (33) is reduced to $P\,\mathrm{d}z^2 = 0$. The condition $\mathrm{d}z = 0$ is associated with the elastic state, while the condition $\mathrm{d}z \neq 0$ is associated with the plastic state. Hence, in the plastic state, P must be equal to zero and eqn (34) leads to

$$(s_{ij} - r_{ij})(s_{ij} - r_{ij}) = (S_y f)^2 \tag{37}$$

where

$$S_y = \left(\sum_{n=1}^{r} \frac{C^n C^n}{N^n}\right)^{-1} \tag{38}$$

and

$$r_{ij} = S_y \sum_{n=1}^{r} \frac{C^n F^n}{N^n} p_{ij}^n. \tag{39}$$

Equation (37) is the von Mises yield criterion with combined isotropic–kinematic hardening, in which f describes the isotropic hardening and r_{ij} the kinematic hardening in the deviatoric behavior. S_y is the initial yield stress. From eqns (30), (32), (38) and (39), the plastic strain increment is now given by

$$\mathrm{d}Q_{ij} = \beta_{ij}\,\mathrm{d}z = \frac{\mathrm{d}z}{S_y}(s_{ij} - r_{ij}) \tag{40}$$

which is normal to the deviatoric yield surface as is seen by comparing with (37).

4. CONCLUDING REMARKS

It has been shown in this paper that, in contrary to Fazio's claim, the limit case of the endochronic theory is capable of describing plastic deformation. The reason that has led to Fazio's erroneous conclusion has been pointed out.

The slope of the stress–strain curve is represented by the function $F(x)$. The form of this function has been numerically investigated for the case of straining, unstraining and cyclic straining. For mathematical simplicity, this calculation has been performed based on the assumption that only one internal state variable is important.

Endochronic equations for the general case using r number of internal variables have also been derived. The derivation is based on the Gibbs formulation. It has been shown that, for the limit case of $k = 1$, the equations reduce to the von Mises yield criterion with combined isotropic–kinematic hardening and that the plastic strain rate obeys the normality rule, in agreement with results obtained by Valanis (1980) using the Helmholtz formulation.

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APPENDIX

Valanis (1975) showed that the Helmholtz and the Gibbs formulations for the endochronic theory are equivalent. Both forms have since been used in numerous applications. In the Helmholtz formulation, the constitutive equation may be expressed in an integral form with the kernel function given by r exponential terms. It has been demonstrated that only one or two exponential terms are sufficient in achieving agreement with experimental data. This simplification is not equivalent though to using only one or two internal variables as in Fazio (1989). Even though it is theoretically admissible, the latter approach may not correctly describe the experimental stress–strain response unless demonstrated. On the other hand in the Gibbs formulation, Wu and Aboutorabi (1988) and Wu *et al.* (1990) have shown that only one or two internal variables are sufficient in obtaining reasonable agreement with experimental results. This is the reason that the Gibbs formulation is used in this paper.

In Fazio (1989), which is based on the Helmholtz formulation, the equation before his (24) should read

$$[E_o f \pm kP\lambda(E_t\varepsilon-\sigma)]\,\mathrm{d}\sigma = E_o[E_o f \pm P\lambda(E_t\varepsilon-\sigma)]\,\mathrm{d}\varepsilon \tag{A1}$$

where λ, P, and E_t have been defined in that paper. This equation was written by Fazio as

$$\mathrm{d}\sigma = E_o\left[\frac{E_o f \pm P\lambda(E_t\varepsilon-\sigma)}{E_o f \pm kP\lambda(E_t\varepsilon-\sigma)}\right]\mathrm{d}\varepsilon. \tag{A2}$$

Note that eqn (A2) is (24) in Fazio (1989). In the case that

$$E_o f \pm kP\lambda(E_t\varepsilon-\sigma) \neq 0, \quad \text{with } k = 1, \tag{A3}$$

(A2) leads to an elastic slope as in Fazio (1989). However, in the case of

$$E_o f \pm kP\lambda(E_t\varepsilon-\sigma) = 0, \quad \text{with } k = 1 \tag{A4}$$

(A2) leads to plastic behavior which was neglected by Fazio. In fact, (A4) can be rewritten as

$$\sigma = E_t\varepsilon \pm \frac{E_o f}{P\lambda} \tag{A5}$$

which describes plastic deformation as predicted by Fazio's model. Equation (A5) does not provide a good description of the experimental stress–strain response due to the reason given in the beginning paragraph of this Appendix. More internal variables are needed in Fazio's approach.

Finally, eqn (60) of Fazio (1989) reads

$$\pm f = \frac{\rho}{E_o} P \sum_{i=1}^{n} B_i\Omega_i(\varepsilon, q^i) \tag{A6}$$

where ρ and B_i are constants defined in that paper. The functions Ω_i are related to the evolution of internal variables q^i. Fazio was correct in pointing out that this equation signifies a constraint condition placed on the endochronic model. In fact, this equation is precisely the constitutive equation which governs the plastic behavior. In the case that the Ω_is are linear in σ and q^i as in eqn (4), (A6) is reduced to (27), which is the condition of

$X = \pm 1$ discussed in Section 2 of this paper. While Fazio recognized the importance of the condition of $X = \pm 1$ in his eqn (60), he did not apply this condition in the discussion of his eqn (24). The $\pm$ sign appearing in (A6) merely distinguishes straining from unstraining. The fact that (27) correctly describes the straining, unstraining, and cyclic behavior is demonstrated in Fig. 2.